

ON TRIVIALITIES OF STIEFEL-WHITNEY CLASSES OF VECTOR BUNDLES OVER ITERATED SUSPENSIONS OF DOLD MANIFOLDS

AJAY SINGH THAKUR

ABSTRACT. A space X is called W -trivial if for every vector bundle ξ over X , the total Stiefel-Whitney class $W(\xi) = 1$. In this article we shall investigate whether the suspensions of Dold manifolds, $\Sigma^k D(m, n)$, is W -trivial or not.

1. INTRODUCTION

Recall [10] that a CW-complex X is said to be W -trivial if for any vector bundle ξ over X , the total Stiefel-Whitney class $W(\xi) = 1$.

It is a theorem of Atiyah-Hirzebruch [1, Theorem 2] that the 9-fold suspension $\Sigma^9 X$ of any CW-complex X is W -trivial (see also [11, Corollary 1.2]). In the same paper, Atiyah-Hirzebruch [1, Theorem 1] have shown that the sphere $S^d = \Sigma^d S^0$ is W -trivial if and only if $d \neq 1, 2, 4$ and 8 (see also [7, Theorem 1]). Here S^0 is the union to two distinct points.

It is therefore an interesting question to understand for what value of $k, 0 \leq k \leq 8$, is the iterated suspension $\Sigma^k X$, of a CW-complex X , W -trivial. Another motivation to study the W -triviality of a CW-complex is its connection with I -triviality [11]. If a CW-complex B is W -trivial then it is I -trivial and hence it satisfies a Borsuk-Ulam type theorem. We refer to [11] and [8] for more details on I -triviality of a CW-complex.

In [9], R. Tanaka obtained results concerning the W -triviality and “ W -triviality except at one dimension” for highly connected CW-complexes. In [11], R. Tanaka determined all pairs (k, n) of positive integers for which $\Sigma^k \mathbb{F}\mathbb{P}^n$ is W -trivial, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} .

In this article we shall investigate when the iterated suspension $\Sigma^k D(m, n)$, of the Dold manifold $D(m, n)$ is W -trivial. Recall [2] that the Dold manifold $D(m, n)$ is an $(m + 2n)$ -dimensional manifold defined as the quotient of $S^m \times \mathbb{C}\mathbb{P}^n$ by the fixed point free involution $(x, z) \mapsto (-x, \bar{z})$. The projection $S^m \times \mathbb{C}\mathbb{P}^n \longrightarrow S^m$ gives rise to a fiber bundle

$$\mathbb{C}\mathbb{P}^n \hookrightarrow D(m, n) \longrightarrow \mathbb{R}\mathbb{P}^m$$

with fiber \mathbb{C}^n and structure group \mathbb{Z}_2 . In particular, we have $D(m, 0) = \mathbb{R}\mathbb{P}^m$ and $D(0, n) = \mathbb{C}\mathbb{P}^n$.

2000 *Mathematics Subject Classification.* 57R20 (55R40, 57R22).

Key words and phrases. Stiefel-Whitney class, Dold manifold, Suspension, K -Theory.

Most of this work was done when the author was a post-doc at Indian Statistical Institute, Bangalore, India.

By the theorem of Atiyah-Hirzebruch [1, Theorem 2], $\Sigma^k D(m, n)$ is W -trivial for $k \geq 9$. So we shall be interested only in the case $0 \leq k \leq 8$ and $m > 0$. We have the following main results.

Theorem 1.1. *Let $\Sigma^k D(m, n)$ be the k -fold suspension of the Dold manifold $D(m, n)$ with $m > 0$. Then $\Sigma^k D(m, n)$ is not W -trivial if*

- (1) $k = 0$.
- (2) $k = 1, 2, 4$ or 8 and $m \geq k$.
- (3) $k = 3, 5$ or 7 and $m + k = 4$ or 8 .
- (4) $k = 6$ and $m = 2$ or 3 .

Theorem 1.2. *Let $\Sigma^k D(m, n)$ be the k -fold suspension of the Dold manifold $D(m, n)$ with $m > 0$ and n even. Then $\Sigma^k D(m, n)$ is W -trivial if*

- (1) $k = 2$ and $m = 1$.
- (2) $k = 3$ and $m \neq 5, 8t + 1$.
- (3) $k = 4$ and $m = 3$.
- (4) $k = 5$ and $m \neq 8t + 3$.
- (5) $k = 6$ and $m \neq 2, 3, 8t + 4$.
- (6) $k = 7$ and $m \neq 1, 8t + 5$.
- (7) $k = 8$ and $m = 1, 2, 3$ or 7 .

We have the following result in the case when n is odd.

Theorem 1.3. *Let $\Sigma^k D(m, n)$ be the k -fold suspension of the Dold manifold $D(m, n)$ with $m > 0$ and n odd. Then $\Sigma^k D(m, n)$ is W -trivial if k and m are as listed above in (1)-(7) of the Theorem 1.2 and any one of the following condition is satisfied:*

- (1) $n + k = 2, 4$ or 8 and $m < k$.
- (2) $n + k = 3, 5$ or 7 and $2n + m + k \neq 4, 8$.
- (3) $n + k = 6$ and $m + n \neq 2, 3$.
- (4) $n + k \geq 9$.

Observe that if we assume $n \geq 3$ in the Theorem 1.3, then $\Sigma^k D(m, n)$ is W -trivial except for $\Sigma^3 D(m, 5)$ with $m \geq 3$ and $\Sigma^5 D(m, 3)$ with $m \geq 5$. In the case when $n = 1$ we have the following theorem.

Theorem 1.4. *Let $\Sigma^k D(m, 1)$ be k -fold suspension of the Dold manifold $D(m, 1)$ with $m > 0$. Then $\Sigma^k D(m, 1)$ is not W -trivial if*

- (1) $k = 1, 3$ or 7 and $m \geq k$.
- (2) $k = 2$ or 4 and $m + k = 2$ or 6 .
- (3) $k = 5$ and $m = 1$ or 2 .

To prove our results we shall require the description, by Fujii-Yasui [6], of KO -groups of Dold manifolds. We shall recall this in Section 2 and prove our results in Section 3.

2. PRELIMINARIES

In this section we shall recall the notations and results from [6], where M. Fujii and T. Yasui have computed the KO -groups of Dold manifolds. These will be used to prove our results.

Let $\pi : D(m, n) \rightarrow D(m, n)/D(m, 0)$ be the projection. Let $p : D(m, n) \rightarrow \mathbb{RP}^m$ be the projection map of the fiber bundle which is described in the introduction and let $i : D(m, 0) \hookrightarrow D(m, n)$ be the inclusion defined by

$$i([x_0, x_1, \dots, x_m]) = [x_0, x_1, \dots, x_m, 1, 0, \dots, 0].$$

Consider the following exact sequence of \widetilde{KO} -groups,

$$\cdots \rightarrow \widetilde{KO}^{-k}(D(m, n)/D(m, 0)) \xrightarrow{\pi^!} \widetilde{KO}^{-k}(D(m, n)) \xrightarrow{i^!} \widetilde{KO}^{-k}(D(m, 0)) \rightarrow \cdots \quad (2.1)$$

Under the identification $D(m, 0) = \mathbb{RP}^m$, we have the composition $i^! \circ p^! = \text{identity}$. Hence the homomorphism

$$p^! : \widetilde{KO}^{-k}(\mathbb{RP}^m) = \widetilde{KO}^{-k}(D(m, 0)) \rightarrow \widetilde{KO}^{-k}(D(m, n))$$

is an injective map and it gives the splitting of the exact sequence (2.1). Let $\widetilde{KO}^{-k}(m, n) := \pi^! \widetilde{KO}^{-k}(D(m, n)/D(m, 0))$. Then we have the following theorem.

Theorem 2.1. [6, Theorem 1] $\widetilde{KO}^{-k}(D(m, n)) = \widetilde{KO}^{-k}(m, n) \oplus p^! \widetilde{KO}^{-k}(\mathbb{RP}^m)$, where $p : D(m, n) \rightarrow \mathbb{RP}^m$ is the natural projection. \square

The KO -groups of the projective space \mathbb{RP}^m has been studied by M. Fujii in [3]. The group $\widetilde{KO}^{-k}(m, n)$ has been computed by M. Fujii and T. Yasui in [6] by making use of the following two homeomorphisms [4, Propsition 2]

- (1) $h_1 : D(m, n)/D(m-1, n) \approx S^m \wedge (\mathbb{CP}^n)^+$.
- (2) $h_2 : D(m, n)/D(m, n-1) \approx S^n \wedge (\mathbb{RP}^{m+n}/\mathbb{RP}^{n-1})$.

Here, for a space X , X^+ denotes the disjoint union of X and a point. The identification of the spaces via homeomorphisms h_1 gives rise to the following long exact sequence [6, p. 58],

$$\begin{aligned} \cdots \longrightarrow \widetilde{KO}^{-k}(S^m \wedge \mathbb{CP}^n) &\xrightarrow{f^!} \widetilde{KO}^{-k}(m, n) \xrightarrow{i^!} \widetilde{KO}^{-k}(m-1, n) \\ &\xrightarrow{\delta} \widetilde{KO}^{-i+1}(S^m \wedge \mathbb{CP}^n) \longrightarrow \cdots, \end{aligned} \quad (2.2)$$

where $f = h_1 \circ \pi$ and $i : D(m-1, n) \hookrightarrow D(m, n)$ is the inclusion. The long exact sequence (2.2) is a direct summand of the following long exact sequence of \widetilde{KO} -groups for the pair $(D(m, n), D(m-1, n))$,

$$\rightarrow \widetilde{KO}^{-k}(D(m, n)/D(m-1, n)) \xrightarrow{\pi^!} \widetilde{KO}^{-k}(D(m, n)) \xrightarrow{i^!} \widetilde{KO}^{-k}(D(m-1, n)) \rightarrow \cdots$$

In the case when $n = 2r$, the groups $\widetilde{KO}^{-k}(m, 2r)$ have been described in Theorem 3 [6]. The proof of the following lemma follows directly from the Theorem 3 of [6] by counting the generators of $\widetilde{KO}^{-k}(m, 2r)$.

Lemma 2.2. Let $\widetilde{KO}^{-k}(m, 2r) := \pi^! \widetilde{KO}^{-k}(D(m, 2r)/D(m, 0))$, where π is the projection. Let $m > 0$. Then $\widetilde{KO}^{-k}(m, 2r) = 0$ if,

- (1) $k = 2$ and $m = 1$.
- (2) $k = 3$ and $m = 8t + 2, 8t + 3, 8t + 4$ or $8t + 6$.
- (3) $k = 5$ and $m = 8t, 8t + 4, 8t + 5$ or $8t + 6$.
- (4) $k = 6$ and $m = 8t + 1, 8t + 5, 8t + 6$ or $8t + 7$.
- (5) $k = 7$ and $m = 8t, 8t + 2, 8t + 6$ or $8t + 7$.

□

In the case when $n = 2r + 1$, the long exact sequence of \widetilde{KO} -groups for the pair $(D(m, 2r + 1), D(m, 2r))$ takes the following form [6, p. 55],

$$\rightarrow \widetilde{KO}^{-k}(D(m, 2r+1)/D(m, 2r)) \rightarrow \widetilde{KO}^{-k}(m, 2r+1) \xrightarrow{i_1^!} \widetilde{KO}^{-k}(m, 2r) \rightarrow, \quad (2.3)$$

where $i_1 : D(m, 2r) \hookrightarrow D(m, 2r + 1)$ is the inclusion. For the long exact sequence (2.3), there exists an algebraic splitting homomorphism

$$\kappa : \widetilde{KO}^{-k}(m, 2r) \rightarrow \widetilde{KO}^{-k}(m, 2r + 1)$$

such that $i_1^! \circ \kappa = \text{identity}$ (refer Section 10 of [6]). In fact, the homomorphism κ is defined as the composition $i_2^! \circ p$, where $i_2 : D(m, 2r + 1) \hookrightarrow D(m, 2r + 2)$ is the inclusion and $p : \widetilde{KO}^{-k}(m, 2r) \rightarrow \widetilde{KO}^{-k}(m, 2r + 2)$ is an algebraic homomorphism defined in Section 10 of [6]. Therefore we have the following theorem.

Theorem 2.3. ([6, Theorem 2])

$$\widetilde{KO}^{-k}(m, 2r + 1) = \widetilde{KO}^{-k}(m, 2r) \oplus \widetilde{KO}^{-k}(D(m, 2r + 1)/D(m, 2r)).$$

□

In the above direct sum decomposition, the group $\widetilde{KO}^{-k}(m, 2r)$ is direct summand of $\widetilde{KO}^{-k}(m, 2r+1)$ via the monomorphism κ , and the group $\widetilde{KO}^{-k}(D(m, 2r+1)/D(m, 2r))$ is isomorphic to $\widetilde{KO}^{-k}(S^{2r+1} \wedge (\mathbb{RP}^{2r+1+m}/\mathbb{RP}^{2r}))$ by the homeomorphism h_2 . The later group, $\widetilde{KO}^{-k}(S^{2r+1} \wedge (\mathbb{RP}^{2r+1+m}/\mathbb{RP}^{2r}))$, has been computed by M. Fujii and T. Yasui in [5].

3. PROOF OF MAIN RESULTS

We first state the following well known facts which we shall use implicitly in our proofs.

- (1) For a vector bundle ξ over a CW-complex X , the smallest integer $k > 0$ with $w_k(\xi) \neq 0$ is a power of 2 (see, for example, [7], p. 94).
- (2) If $\widetilde{KO}(X) = 0$ then every vector bundle over X is stably trivial and hence $W(\xi) = 1$ for any vector bundle ξ over X . Thus X is W -trivial.
- (3) Recall [2] that the \mathbb{Z}_2 -cohomology ring of the Dold manifold $D(m, n)$ is given as

$$H^*(D(m, n); \mathbb{Z}_2) = \mathbb{Z}_2[c, d]/(c^{m+1} = 0, d^{n+1} = 0),$$

where $c \in H^1(D(m, n); \mathbb{Z}_2)$ and $d \in H^2(D(m, n); \mathbb{Z}_2)$. If c' is the generator of $H^1(\mathbb{RP}^m; \mathbb{Z}_2)$ and d' is the generator of $H^2(\mathbb{CP}^n; \mathbb{Z}_2)$ then $p^*(c') = c$ and $i^*(d) = d'$, where $p : D(m, n) \rightarrow \mathbb{RP}^m$ is the projection and $i : \mathbb{CP}^n \hookrightarrow D(m, n)$ is the fibre inclusion of the fibre bundle

$$\mathbb{CP}^n \xrightarrow{i} D(m, n) \xrightarrow{p} \mathbb{RP}^m.$$

The action of Steenrod squares on the cohomology ring $H^*(D(m, n); \mathbb{Z}_2)$ are completely determined by the fact that $Sq^1 d = cd$.

Proof of Theorem 1.1. Consider the projection map $p : D(m, n) \rightarrow \mathbb{RP}^m$. Since the composition map,

$$\mathbb{RP}^m = D(m, 0) \hookrightarrow D(m, n) \xrightarrow{p} \mathbb{RP}^m$$

is the identity map, the induced map

$$p^* : H^i(\mathbb{RP}^m; \mathbb{Z}_2) \rightarrow H^i(D(m, n); \mathbb{Z}_2)$$

is injective. Hence the suspension map induces an injective map

$$\Sigma^k p^* : H^i(\Sigma^k \mathbb{RP}^m; \mathbb{Z}_2) \rightarrow H^i(\Sigma^k D(m, n); \mathbb{Z}_2).$$

Now if there is a vector bundle ξ over $\Sigma^k \mathbb{RP}^m$ with $w_i(\xi) \neq 0$, then $w_i(p^*(\xi)) \neq 0$. Thus we have showed that if $\Sigma^k \mathbb{RP}^m$ is not W -trivial then the k -fold suspension $\Sigma^k D(m, n)$ of Dold manifold $D(m, n)$ is also not W -trivial. Thus the proof of the Theorem 1.1, now follows from the Theorem 1.4 [11]. \square

We now come to the proof of the Theorem 1.2. The Theorem 1.2 will be proved in sequence of propositions below.

Proposition 3.1. *Let $m > 0$. Then $\Sigma^k D(m, 2r)$ is W -trivial if*

- (1) $k = 2$ and $m = 1$.
- (2) $k = 3$ and $m = 8t + 2, 8t + 3, 8t + 4$ or $8t + 6$.
- (3) $k = 5$ and $m = 8t, 8t + 4, 8t + 5$ or $8t + 6$.
- (4) $k = 6$ and $m = 8t + 1, 8t + 5, 8t + 6$ or $8t + 7$.
- (5) $k = 7$ and $m = 8t, 8t + 2, 8t + 6$ or $8t + 7$.

Proof. Let k and m be as in the statement of the proposition. Note that the group $\widetilde{KO}^{-k}(m, 2r) = 0$ (Lemma 2.2) and the k -fold suspension $\Sigma^k \mathbb{RP}^m$ of \mathbb{RP}^m is W -trivial (Theorem 1.4 [11]). Now by the decomposition,

$$\widetilde{KO}^{-k}(D(m, 2r)) = \widetilde{KO}^{-k}(m, 2r) \oplus p^! \widetilde{KO}^{-k}(\mathbb{RP}^m),$$

of the Theorem 2.1, any vector bundle $\xi \in \widetilde{KO}^{-k}(D(m, 2r))$ is stably equivalent to $\eta \oplus \nu$, where $\eta \in \widetilde{KO}^{-k}(m, 2r)$ and $\nu \in p^! \widetilde{KO}^{-k}(\mathbb{RP}^m)$. Since $W(\eta) = 1$ and $W(\nu) = 1$, we have $W(\xi) = 1$. This completes the proof of the proposition. \square

Proposition 3.2. *Let $m > 0$. Then $\Sigma^k D(m, 2r)$ is W -trivial if*

- (1) $k = 3$ and $m = 8t + 5 (t > 0)$ or $8t + 7$.
- (2) $k = 5$ and $m = 8t + 1$ or $8t + 7$.
- (3) $k = 6$ and $m = 8t$ or $8t + 2 (t > 0)$.
- (4) $k = 7$ and $m = 8t + 1$ or $8t + 3$.

Proof. Let k and m be as in the statement of the proposition. Then consider the following exact sequence, (2.2),

$$\cdots \rightarrow \widetilde{KO}^{-k}(S^m \wedge \mathbb{CP}^{2r}) \xrightarrow{f^!} \widetilde{KO}^{-k}(m, 2r) \xrightarrow{i^!} \widetilde{KO}^{-k}(m-1, 2r) \rightarrow \cdots$$

Since $\widetilde{KO}^{-k}(m-1, 2r) = 0$ (Lemma 2.2), the map $f^!$ is surjective. By Theorem 1.5 [11], $\Sigma^{k+m} \mathbb{CP}^{2r}$ is W -trivial and this implies that for any vector bundle $\xi \in \widetilde{KO}^{-k}(m, 2r)$ the total Stiefel-Whitney class $W(\xi) = 1$. Furthermore, we know that $\Sigma^k \mathbb{RP}^m$ is W -trivial (Theorem 1.4 [11]). Hence by the decomposition

$$\widetilde{KO}^{-k}(D(m, 2r)) = \widetilde{KO}^{-k}(m, 2r) \oplus p^! \widetilde{KO}^{-k}(\mathbb{RP}^m),$$

of the Theorem 2.1, we conclude that $\Sigma^k D(m, 2r)$ is W -trivial. \square

Proposition 3.3. *Let $m > 0$. Then $\Sigma^k D(m, 2r)$ is W -trivial if*

- (1) $k = 3$ and $m = 8t$.
- (2) $k = 5$ and $m = 8t + 2$.
- (3) $k = 6$ and $m = 8t + 3$.
- (4) $k = 7$ and $m = 8t + 4$.
- (5) $k = 8$ and $m = 1$.

Proof. Let k and m be as in the statement of the proposition. Consider the inclusion,

$$i : D(m-1, 2r) \hookrightarrow D(m, 2r)$$

Note that the quotient $D(m, 2r)/D(m-1, 2r) \approx \Sigma^m \mathbb{CP}^{2r} \vee S^m$, and hence the induced map

$$i^* : H^p(D(m, 2r); \mathbb{Z}_2) \rightarrow H^p(D(m-1, 2r); \mathbb{Z}_2)$$

is monomorphism when $p + m$ is odd. Thus the induced suspension map

$$\Sigma^q i^* : H^q(\Sigma^k D(m, 2r); \mathbb{Z}_2) \rightarrow H^q(\Sigma^k D(m-1, 2r); \mathbb{Z}_2)$$

is monomorphism for even q .

Now, if there is a vector bundle ξ over $\Sigma^k D(m, 2r)$ with $w_{2^s}(\xi) \neq 0$ for some $s > 0$, then $w_{2^s}(i^*(\xi)) \neq 0$. But we know that $\Sigma^k D(m-1, 2r)$ is W -trivial (refer Proposition 3.2 for $k \neq 8$ and Theorem 1.5 [11] for $k = 8$). This gives a contradiction and thus we conclude that $\Sigma^k D(m, 2r)$ is W -trivial. \square

Proposition 3.4. *The iterated suspensions $\Sigma^8 D(2, 2r)$ and $\Sigma^8 D(3, 2r)$ are W -trivial.*

Proof. We shall first prove that $\Sigma^8 D(2, 2r)$ is W -trivial. Let ξ be a vector bundle over $\Sigma^8 D(2, 2r)$. Let $s \geq 4$ be such that $2^s \leq \dim(\Sigma^8 D(2, 2r))$ and $w_j(\xi) = 0$ for $0 < j < 2^s$. We shall show that $w_{2^s}(\xi) = 0$ and thus this will imply that $\Sigma^8 D(2, 2r)$ is W -trivial.

Let $a \in H^{2^s-8}(D(2, 2r); \mathbb{Z}_2)$ be the cohomology class which maps to $w_{2^s}(\xi) \in H^{2^s}(\Sigma^8 D(2, 2r); \mathbb{Z}_2)$ under the suspension isomorphism

$$H^{2^s-8}(D(2, 2r); \mathbb{Z}_2) \rightarrow H^{2^s}(\Sigma^8 D(2, 2r); \mathbb{Z}_2).$$

By Lemma 3.3 of [11] and the fact that the Steenrod squares commutes with suspension homomorphism we have

$$Sq^i(a) = 0 \text{ for all } 0 < i < 2^{s-1}.$$

Now observe that the vector space $H^{2^s-8}(D(2, 2r); \mathbb{Z}_2)$ is generated by

$$d^{2^{s-1}-4} \text{ and } c^2 d^{2^{s-1}-5}.$$

Therefore,

$$a = x \cdot d^{2^{s-1}} + y \cdot c^2 d^{2^{s-1}}$$

for $x, y \in \mathbb{Z}_2$. If $x \neq 0$ then $w_{2^s}(i^*\xi) \neq 0$, where

$$i : \Sigma^8 \mathbb{CP}^{2r} \hookrightarrow \Sigma^8 D(2, 2r)$$

is the inclusion map and $i^*\xi$ is the pullback bundle over $\Sigma^8 \mathbb{CP}^{2r}$. But since $\Sigma^8 \mathbb{CP}^{2r}$ is W -trivial (Theorem 1.5 [11]), we have a contradiction and hence $x = 0$. Further, since

$$Sq^2(c^2 d^{2^{s-1}}) = c^2 d^{2^{s-1}-4} \neq 0,$$

we have $y = 0$. Hence $w_{2^s}(\xi) = 0$. This completes the proof of W -triviality of $\Sigma^8 D(2, 2r)$.

The proof of W -triviality of $\Sigma^8 D(3, 2r)$ proceeds along the same lines as the proof of Proposition 3.3 using the fact that $\Sigma^8 D(2, 2r)$ is W -trivial. \square

In the following proposition, n can be both even or odd.

Proposition 3.5. *The iterated suspensions $\Sigma^4 D(3, n)$ and $\Sigma^8 D(7, n)$ are W -trivial.*

Proof. Let $s \geq 3$ be such that $2^s \leq \dim(\Sigma^4 D(3, n))$. Observe that the vector space $H^{2^s-4}(P(3, n); \mathbb{Z}_2)$ is generated by

$$d^{2^{s-1}-2} \text{ and } c^2 d^{2^{s-1}-3}.$$

Here note that $d^{2^{s-1}-2}$ will be zero if $n > 2^s - 4$. Further observe that

$$Sq^1(c^2 d^{2^{s-1}-3}) = c^3 d^{2^{s-1}-3} \neq 0.$$

With these observations the proof of the W -triviality of $\Sigma^4 D(3, n)$ proceeds along the same lines as the proof of W -triviality of $\Sigma^8 D(2, 2r)$ in the Proposition 3.4.

Similarly we can argue that $\Sigma^8 D(7, n)$ is W -trivial. Here we need to observe that, for $s \geq 4$ such that $2^s \leq \dim(\Sigma^8 D(7, n))$, the vector space $H^{2^s-8}(P(7, n); \mathbb{Z}_2)$ is generated by

$$d^{2^{s-1}-4}, c^2 d^{2^{s-1}-5}, c^4 d^{2^{s-1}-6} \text{ and } c^6 d^{2^{s-1}-7}.$$

Here again some of these cohomology classes, except $c^6 d^{2^{s-1}-7}$, can be zero. Further observe that $Sq^1(c^2 d^{2^{s-1}-5}) = c^3 d^{2^{s-1}-5}$, $Sq^1(c^6 d^{2^{s-1}-7}) = c^7 d^{2^{s-1}-7}$ and $Sq^1(c^4 d^{2^{s-1}-6}) = 0$ but $Sq^2(c^4 d^{2^{s-1}-6}) = c^6 d^{2^{s-1}-6}$. Now the proof of W -triviality of $\Sigma^8 D(7, n)$ will proceed along the same lines as in the case $\Sigma^4 D(3, n)$. This completes the proof of the proposition. \square

Remark 3.6. More generally one can prove that the m -fold suspension $\Sigma^m D(m-1, n)$ of the Dold manifold $D(m-1, n)$ with $m > 0$ is W -trivial by the method used in proving the Proposition 3.5.

The proof of the Theorem 1.2 follows from Propositions 3.1, 3.2, 3.3, 3.4 and 3.5.

We now come to the proof of the Theorem 1.3. First we make the following observation concerning the W -triviality of stunted projective space.

Lemma 3.7. *Let $\mathbb{RP}^m/\mathbb{RP}^n$ be the stunted projective space with $m \geq n$. Then $\Sigma^k(\mathbb{RP}^m/\mathbb{RP}^n)$ is W -trivial if*

- (1) $k = 1, 2, 4$ or 8 and $m < k$.
- (2) $k = 3, 5$ or 7 and $m + k \neq 4, 8$.
- (3) $k = 6$ and $m \neq 2, 3$.
- (4) $k \geq 9$.

Proof. Let $X = \mathbb{RP}^m/\mathbb{RP}^n$. Let $\alpha : \mathbb{RP}^m \rightarrow X$ be the projection map. Then the induced suspension homomorphism

$$(\Sigma^k \alpha)^* : H^i(\Sigma^k X; \mathbb{Z}_2) \rightarrow H^i(\Sigma^k \mathbb{RP}^m; \mathbb{Z}_2),$$

is an isomorphism for $i > n + k$. Hence if there is a vector bundle ξ over $\Sigma^k X$ with $w_i(\xi) \neq 0$ then $w_i((\Sigma^k \alpha)^* \xi) \neq 0$. Thus the W -triviality of $\Sigma^k \mathbb{RP}^m$ implies the W -triviality of $\Sigma^k X$. Now the proof of the lemma follows from the Theorem 1.4 [11]. \square

Proposition 3.8. *Let $\Sigma^k D(m, 2r + 1)$ be k -fold suspension of the Dold manifold $D(m, 2r + 1)$. Then $\Sigma^k D(m, 2r + 1)$ is W -trivial if $\Sigma^k D(m, 2r + 2)$ and $\Sigma^{2r+1+k}(\mathbb{RP}^{2r+1+m}/\mathbb{RP}^{2r})$ are W -trivial.*

Proof. Let ξ be a vector bundle over $\Sigma^k D(m, 2r + 1)$. We shall prove the proposition by showing that the total Stiefel-Whitney class $W(\xi) = 1$.

Consider the following decomposition by Theorem 2.1,

$$\widetilde{KO}^{-k}(D(m, 2r + 1)) = \widetilde{KO}^{-k}(m, 2r + 1) \oplus \widetilde{KO}^{-k}(\mathbb{RP}^m).$$

Since the W -triviality of $\Sigma^k D(m, 2r + 2)$ implies the W -triviality of $\Sigma^k \mathbb{RP}^m$ (Theorem 2.1), we can assume that $\xi \in \widetilde{KO}^{-k}(m, 2r + 1)$. Now, since the $(2r + 1 + k)$ -fold suspension $\Sigma^{2r+1+k}(\mathbb{RP}^{2r+1+m}/\mathbb{RP}^{2r})$ is W -trivial and by the decomposition,

$$\widetilde{KO}^{-k}(m, 2r + 1) = \widetilde{KO}^{-k}(m, 2r) \oplus \widetilde{KO}^{-k}(\Sigma^{2r+1}(\mathbb{RP}^{2r+1+m}/\mathbb{RP}^{2r})),$$

of the Theorem 2.3, we can further assume that $\xi = \kappa(\gamma)$ for some $\gamma \in \widetilde{KO}^{-k}(m, 2r)$. Here κ is the monomorphism with respect to which $\widetilde{KO}^{-k}(m, 2r)$ is direct summand of $\widetilde{KO}^{-k}(m, 2r + 1)$ (refer to section 10 of [6] for more details). By the definition of κ , we have $\xi = i_2^!(\eta)$ for some $\eta \in \widetilde{KO}^{-k}(m, 2r + 2)$. Here $i_2 : D(m, 2r + 1) \hookrightarrow D(m, 2r + 2)$ is the inclusion. Thus, by the W -triviality of $\Sigma^k D(m, 2r + 2)$, we conclude that $W(\xi) = 1$. This completes the proof of the proposition. \square

Now, under the given hypothesis on $k, n = 2r + 1$ and m , as in statement of the Theorem 1.3, the k -fold suspension $\Sigma^k D(m, 2r + 2)$ is W -trivial for all $r \geq 0$ (Theorem 1.2). It is also clear by the Lemma 3.7 that for these values of $k, n = 2r + 1$ and m , the space $\Sigma^{2r+1+k}(\mathbb{RP}^{2r+1+m}/\mathbb{RP}^{2r})$ is W -trivial. Thus the proof of the Theorem 1.3 follows from the Proposition 3.8.

We now come to the proof of the Theorem 1.4. First note the following lemma.

Lemma 3.9. *The k -fold suspension $\Sigma^k D(m, n)$ is not W -trivial if the $(n + k)$ -suspension $\Sigma^{n+k}(\mathbb{RP}^{n+m}/\mathbb{RP}^{n-1})$ is not W -trivial.*

Proof. Observe that the inclusion $i : D(m, n - 1) \hookrightarrow D(m, n)$ induces a surjective map

$$i^* : H^p(D(m, n); \mathbb{Z}_2) \rightarrow H^p(D(m, n - 1); \mathbb{Z}_2), \text{ for all } p,$$

in cohomology group. Thus

$$\pi^* : H^p(D(m, n)/D(m, n - 1); \mathbb{Z}_2) \rightarrow H^p(D(m, n); \mathbb{Z}_2), \text{ for } p > 0,$$

is injective. Here

$$\pi : D(m, n) \rightarrow D(m, n)/D(m, n - 1) \approx \Sigma^n(\mathbb{RP}^{n+m}/\mathbb{RP}^{n-1})$$

is the quotient map. Thus the induced suspension morphism

$$\Sigma^k \pi^* : H^p(\Sigma^{n+k}(\mathbb{RP}^{n+m}/\mathbb{RP}^{n-1}); \mathbb{Z}_2) \rightarrow H^p(\Sigma^k D(m, n); \mathbb{Z}_2)$$

is injective for $p > 0$. Hence if there is a vector bundle ξ over $\Sigma^{n+k}(\mathbb{RP}^{n+m}/\mathbb{RP}^{n-1})$ with $W(\xi) \neq 1$, then $W(\pi^*(\xi)) \neq 1$. This completes the proof of the Lemma. \square

Now the proof of the Theorem 1.4 follows immediately from Lemma 3.9 and Theorem 1.4 [11].

This completes the proof of the results stated in the introduction. As noted earlier we still do not know whether $\Sigma^3 D(m, 5)$ and $\Sigma^5 D(m, 3)$ are W -trivial or not

for all m . Though this can be answered in few cases, we do not have a complete picture. For example if $m = 2, 3$ or 4 then $\Sigma^3 D(m, 5)$ is W -trivial by Proposition 3.8 and Corollary 1.3 of [11]. By similar argument we can say that if $m = 1, 2, 4, 5$ or 6 then $\Sigma^5 D(m, 3)$ is W -trivial. We also have the following proposition.

Proposition 3.10. *Let $n > 1$ and $n \not\equiv 3 \pmod{4}$. Then $\Sigma^4 D(1, n)$ is not W -trivial.*

Proof. Consider the following long exact sequence, (2.2),

$$\cdots \rightarrow \widetilde{KO}^{-4}(\Sigma \mathbb{CP}^n) \xrightarrow{f^!} \widetilde{KO}^{-4}(1, n) \xrightarrow{i^!} \widetilde{KO}^{-4}(0, n) \xrightarrow{\delta} \widetilde{KO}^{-3}(\Sigma \mathbb{CP}^n) \rightarrow \cdots$$

Since $\widetilde{KO}^{-4}(\Sigma \mathbb{CP}^n) = 0$ [3, Theorem 2], the homomorphism

$$i^! : \widetilde{KO}^{-4}(1, n) \longrightarrow \widetilde{KO}^{-4}(0, n)$$

is a monomorphism. We shall first prove that $i^!$ is an isomorphism.

Depending upon whether n is odd or even, we write $n = 2r$ or $2r + 1$. As $n \not\equiv 3 \pmod{4}$ we have, by Theorem 2 [3],

$$\widetilde{KO}^{-4}(\mathbb{CP}^n) = \widetilde{KO}^{-4}(0, n) = \mathbb{Z}^r.$$

Now if $n = 2r + 1$ then we have the decomposition,

$$\widetilde{KO}^{-4}(1, 2r + 1) = \widetilde{KO}^{-4}(1, 2r) \oplus \widetilde{KO}^{-4}(\Sigma^{2r+1}(\mathbb{RP}^{2r+2}/\mathbb{RP}^{2r})),$$

by the Theorem 2.3. We have $\widetilde{KO}^{-4}(\Sigma^{2r+1}(\mathbb{RP}^{2r+2}/\mathbb{RP}^{2r})) = 0$ (refer Table (2) on p. 47 of [5]) and $\widetilde{KO}^{-4}(1, 2r) = \mathbb{Z}^r$ [6, Theorem 3]. Therefore,

$$\widetilde{KO}^{-4}(1, n) = \mathbb{Z}^r.$$

Now putting all these values of \widetilde{KO} -groups in the above long exact sequence one can easily conclude that the homomorphism

$$\delta : \widetilde{KO}^{-4}(0, n) \longrightarrow \widetilde{KO}^{-3}(\Sigma \mathbb{CP}^n) = \widetilde{KO}^{-4}(\mathbb{CP}^n)$$

is zero. Thus the homomorphism $i^!$ is an isomorphism.

As $\Sigma^4 \mathbb{CP}^n$ is not W -trivial (Theorem 1.5 [11]), there is a vector bundle $\xi \in \widetilde{KO}^{-4}(0, n) = \widetilde{KO}^{-4}(\mathbb{CP}^n)$ with $W(\xi) \neq 1$. Thus there is a vector bundle over $\Sigma^4 D(1, n)$ with non-trivial Stiefel-Whitney class. This completes the proof of the proposition. \square

Remark 3.11. Note that there is no integer s such that $5 \leq 2^s \leq \dim(\Sigma^4 D(1, 1)) = 7$. Hence $\Sigma^4 D(1, 1)$ is W -trivial.

Following are the cases which we have not been able to settle: (i) $k = 3$ and $m = 8t + 1$ (ii) $k = 4$ and $m = 2$ (iii) $k = 5$ and $m = 8t + 3$ (iv) $k = 6$ and $m = 8t + 4$ (v) $k = 7$ and $m = 8t + 5$ (vi) $k = 8$ and $m = 4, 5$ or 6 . In addition to these cases, we also do not know whether $\Sigma^k D(m, n)$ is W -trivial or not when k , m and n satisfy any one of the following condition: (i) $k = 3$, $n = 5$ and $m \geq 5$ (ii) $k = 5$, $n = 3$ and $m \geq 7$. (iii) $k = 4$, $m = 1$ and $n \equiv 3 \pmod{4}$.

Acknowledgement. The author is grateful to Aniruddha Naolekar for several valuable discussions. He also thanks him for his helpful comments and suggestions on the manuscript.

REFERENCES

- [1] M. F. Atiyah and F. Hirzebruch, *Bott periodicity and the parallelizability of the spheres*, Proc. Cambridge Philos. Soc. **57** (1961), 223-226.
- [2] A. Dold, *Erzeugende der Thomschen Algebra* \mathfrak{N} , Math. Z. **65** (1956), 25-35.
- [3] M. Fujii, *KO-groups of projective spaces*, Osaka J. Math. **4** (1967), 141-149.
- [4] M. Fujii, *Ring structures of KU-cohomologies of Dold manifolds*, Osaka J. Math. **6** (1969), 107-115.
- [5] M. Fujii and T. Yasui, *KO-groups of the stunted real projective spaces*, Math. J. Okayama Univ. **16** (1973), 47-54.
- [6] M. Fujii and T. Yasui, *KO-cohomologies of the Dold manifolds*, Math. J. Okayama Univ. **16** (1973), 55-84.
- [7] J. Milnor and J. D. Stasheff, *Characteristic classes* Annals of Mathematics Studies **76** Princeton University Press, Princeton, 1974.
- [8] R. Tanaka, *On the index and co-index of sphere bundle*, Kyushu J. Math. **57** (2003), 371-382.
- [9] R. Tanaka, *On trivialities of Stiefel-Whitney classes of vector bundles over highly connected complexes*, Topology and its Applications **155** (2008), 1687-1693.
- [10] R. Tanaka, *On trivialities of Stiefel-Whitney classes of vector bundles over iterated suspension spaces*, Homology, Homotopy Appl. **12** (2010), 357-366.
- [11] R. Tanaka, *A Borsuk-ulam type theorem over iterated suspensions of real projective spaces*, J. Korean Math. Soc. **49** (2012), 251-263.

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, COLABA, MUMBAI 400 005, INDIA

E-mail address: `athakur@math.tifr.res.in`